Giant gravitons and the supersymmetric states of $\mathcal{N} = 4$ Yang-Mills

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based on hep-th/0606087 with Indranil Biswas, Davide Gaiotto and Shiraz Minwalla

Introduction

We want to classify the supersymmetric states of $\mathcal{N} = 4$ super Yang-Mills living on $S^3\times$Time.

Classifying the 1/16 BPS states would tell us something about the supersymmetric black holes in $AdS_5 \times S^5$.

As we've had no success with this, we will try to develop some tricks for 1/8 BPS states that will hopefully generalise.

Outline

- Supersymmetric states and black holes
- Giant gravitons
- Quantisation
- Conclusions and future directions

Counting states

We would like to classify the states of the theory using the bosonic symmetries: The SO(2,4) conformal group and SO(6) R-symmetries.

i.e. count the number of states for each value of the Noether charges.

This can be summarised in a partition function:

$$Z = \mathrm{Tr}\, \mathrm{e}^{\mu_i Q_i}$$

Due to the state-operator correspondence for a conformal theory on S^3, we can count operators instead.

Supersymmetric spectrum

Supersymmetric states lie in short representations of the superconformal algebra.
SUSY $\rightarrow$ non-SUSY requires short reps joining to form a long rep.

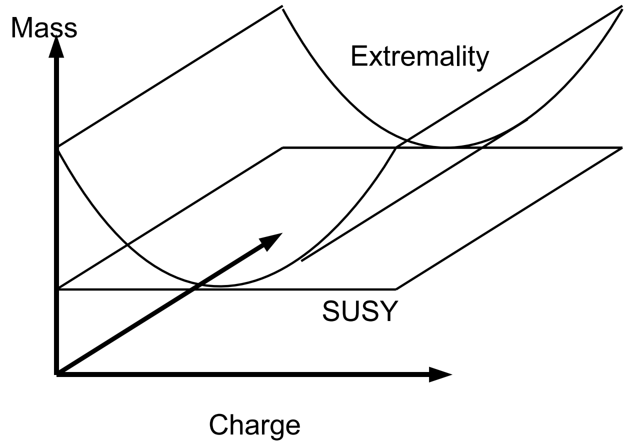
We expect that this will only happen at special points - e.g. $\lambda = 0$, so the SUSY spectrum could match between small and large coupling.

AdS$_5$ black holes

A general black hole has six parameters $(\Delta, J, \bar{J}, R_1, R_2, R_3)$.

For each value of $(J, \bar{J}, R_1, R_2, R_3)$, there is an extremal black hole.

If these five charges satisfy an additional relation, this black hole preserves 1/16 of the supersymmetries.



[Gutowski,Reall; Chong et al.; Kunduri et al.]

1/16 BPS states

At zero coupling:

- Qualitative but not quantitative matching
- No sign of relation between charges

At weak coupling:

- Haven't found the spectrum
- Computing an index doesn't work

[Kinney,Maldacena,Minwalla,Raju]

Let's look at the easier 1/8 BPS states and learn some tricks to help with the 1/16 problem.

1/8 BPS states

These are invariant under both components of Q^1_α and their complex conjugates $S_{1\,\alpha}$.

They are in 1-1 correspondence with cohomology classes of Q^1_α, i.e.

$$Q\,|\psi\rangle = 0$$

$$|\psi\rangle \sim |\psi\rangle + Q\,|\phi\rangle$$

Each cohomology class contains one state that is also annihilated by S.

We can count operators instead of states.

1/8 BPS states

The supercharge acts (in $\mathcal{N}=1$ language) as:

$$Q_\alpha \bar{\phi}^m = 0\,, \qquad Q_\alpha \phi_m = \psi_{m\alpha}\,,$$

$$Q_\alpha \psi_{m\beta} = g_{YM}\epsilon_{\alpha\beta}\epsilon_{mkl}[\bar{\phi}^k, \bar{\phi}^l]\,, \qquad Q_\alpha \lambda_\beta = f_{\alpha\beta} + g_{YM}\epsilon_{\alpha\beta}[\phi_m, \bar{\phi}^m]\,,$$

$$Q_\alpha \bar{\psi}^m_{\dot\beta} = D_{\alpha\dot\beta}\bar{\phi}^m\,, \qquad Q_\alpha \bar{\lambda}_{\dot\beta} = 0\,,$$

$$Q_\alpha A_{\beta\dot\gamma} = \epsilon_{\alpha\beta}\bar{\lambda}_{\dot\gamma}\,. \qquad \Longrightarrow Q_\alpha D_{\beta\dot\gamma} = g_{YM}\epsilon_{\alpha\beta}[\bar{\lambda}_{\dot\gamma},\ \ ]\,.$$

The Q-closed letters are $\bar{\phi}^m$ and $\bar{\lambda}_{\dot\beta}$. Their commutators are Q-exact. They are simultaneously diagonalisable in cohomology.

They can be counted in terms of eigenvalues.

3D Harmonic oscillator

The states built out of the scalars can be thought of as N bosons moving in a three dimensional harmonic oscillator.

$(\bar{\phi}^1_a)^{n_1}(\bar{\phi}^2_a)^{n_2}(\bar{\phi}^3_a)^{n_3}$ maps onto boson number a in the state $|n_1, n_2, n_3\rangle$.

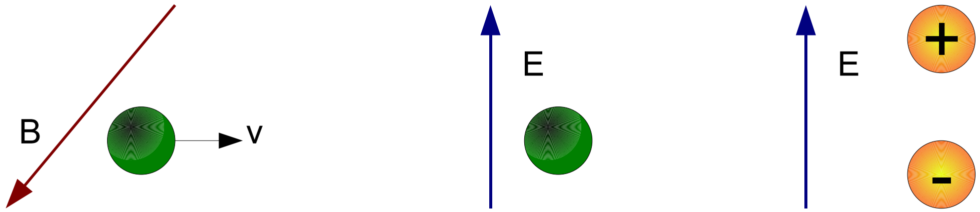
Bosons because permutations are part of the gauge invariance.

SO(6) charges are the total excitation $\#$'s of each oscillator.

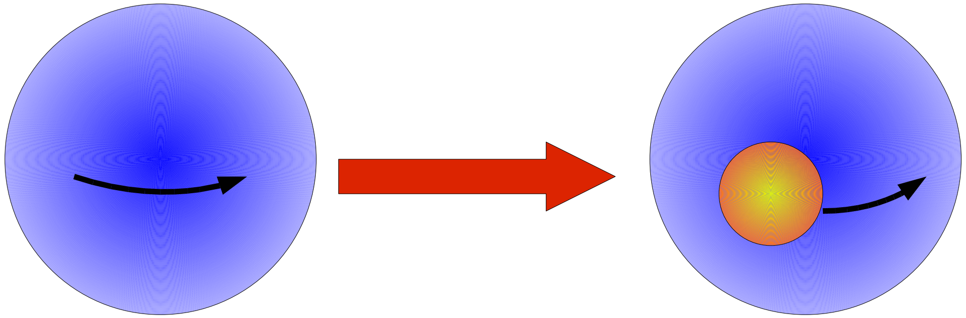
Energies $<< N$ described by multi-gravitons.
Energies $\sim N$ described by giant gravitons.

Giant gravitons

Analogy: neutral particle moving in a magnetic field.



Graviton moving with high angular momentum around S^5



Likes to puff out into a D3 brane.

[McGreevy et al.; Grisaru et al.; Hashimoto et al.]

Quantising these produces finite N, 1/2 BPS spectrum.

1/8 BPS giant gravitons

Mikhailov's construction:

- Embed S^5 in $\mathbb{C}^3$

$$|x|^2 + |y|^2 + |z|^2 = 1$$

- Pick a holomorphic function $f(x, y, z)$
- The D3-brane wraps the intersection of the surface $f(\mathrm{e}^{-\mathrm{i}t}x, \mathrm{e}^{-\mathrm{i}t}y, \mathrm{e}^{-\mathrm{i}t}z) = 0$ with the S^5.

It can be shown that this preserves 1/8 of the supersymmetries.

Doesn't include worldvolume gauge fields and fermions.

Quantisation

First we need to describe the system in the Hamiltonian formalism.

We need a phase space and a Poisson bracket:

$$\{x^i, x^j\} = \omega^{ij}\,, \qquad \{f, g\} = \omega^{ij}(\partial_i f)(\partial_j g)\,,$$

or, equivalently, a symplectic form: $\omega_{ij} = \left[\omega^{ij}\right]^{-1}$.

Then we can use the standard procedure of Geometric Quantisation.

Crnkovic-Witten-Zuckerman formalism

We can identify the phase space with the space of solutions to the equations of motion.

We then find ω by plugging the solutions into:

$$\omega = \int \mathrm{d}x\, \delta\left(\frac{\partial L}{\partial \dot{\phi}^i}\right) \wedge \delta\phi^i$$

where ϕ^i are the dynamical fields.

We will apply this to Mikhailov's solutions with the Born-Infeld action.

Mikhailov's phase space

Solutions parameterised by one holomorphic function.

This is an infinite dimensional space. We regulate it by restricting to polynomials made from a finite number of monomials:

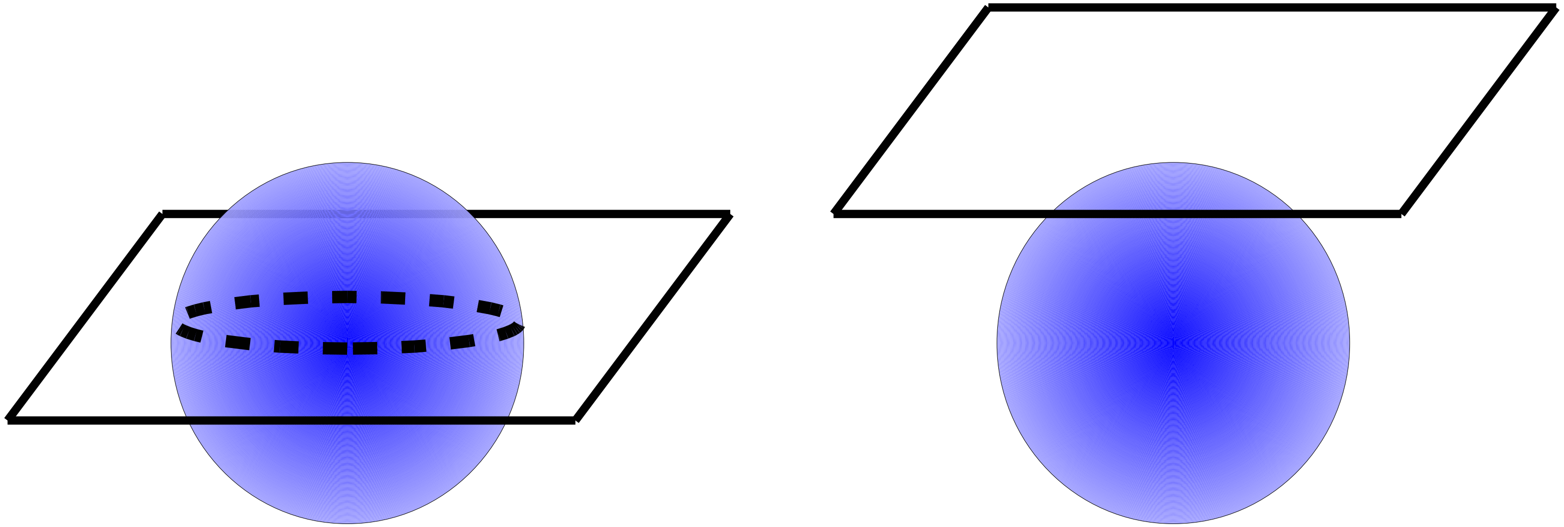
$$f(z_1, z_2, z_3) = \sum_{\vec{n} \in C} c_{\vec{n}} \, (z^1)^{n_1} (z^2)^{n_2} (z^3)^{n_3} \, .$$

$c_{\vec{n}}$ and $\lambda c_{\vec{n}}$ describe the same surface.

It looks like $\mathbb{CP}^{n_C - 1}$.

But unfortunately . . .

. . . not all surfaces touch the sphere:



Eats holes out of phase space.

However, everything would work perfectly if we ignored these problems. . . [Beasley]

Geometric quantisation of $\mathbb{CP}^n$

$\mathbb{CP}^n$ has a canonical two-form (Fubini-Study):

$$\omega_{\mathrm{FS}} = \frac{1}{4\pi \mathrm{i}} \frac{1}{|z|^2} \left[\mathrm{d}\bar{z}^i - \frac{\bar{z}^i z^j}{|z|^2} \mathrm{d}\bar{z}^j \right] \wedge \left[\mathrm{d}z^i - \frac{z^i \bar{z}^j}{|z|^2} \mathrm{d}z^j \right] .$$

Suppose that our symplectic form is in the cohomology class $(2\pi N)[\omega_{\mathrm{FS}}]$.

It is a standard result that the Hilbert space is the space of degree N homogeneous polynomials in the z^i.

3D harmonic oscillator (again)

We can map this to the 3D harmonic oscillator as follows:

$c_{\vec{n}} \to a^\dagger_{\vec{n}}$: the creation operator for a particle in the state $|n_1, n_2, n_3\rangle$.

A monomial of degree N acting on the vacuum produces a N-particle state.

These states transform the same way under U(3) as our 3D harmonic oscillator.

Problems

- Some surfaces do not touch the sphere, e.g.

$$c_i z^i - 1 = 0 \quad \text{for} \quad |c|^2 < 1\,.$$

- Singularities when the function factorises - the surface degenerates, e.g.

$$x^2 + y^2 + \epsilon z^2 = 0 \quad \text{as} \quad \epsilon \longrightarrow 0\,.$$

- Finding the cohomology class of ω.

Resolution

- There is a U(3) invariant coordinate change that maps the holes to a point leaving the phase space topologically $\mathbb{CP}^{n_C-1}$
- There is a geometric description of the symplectic form in terms of volumes swept out by deformations that shows that any singularities are mild enough to allow geometric quantisation.
- The structure of $\mathbb{CP}^{n_C-1}$ means that the cohomology class is the same for all sets of monomials and we can show that it is $2\pi N$ for linear functions.

Example: Linear functions

Let's look at the space of functions $f(z^i) = c_i z^i - 1$.

The symplectic form is:

$$\omega = 2N\left[\left(\frac{1}{|c|^2} - \frac{1}{|c|^4}\right)\frac{\mathrm{d}\bar{c}^i \wedge \mathrm{d}c_i}{2\mathrm{i}} - \left(\frac{1}{|c|^2} - \frac{2}{|c|^4}\right)\frac{\bar{c}^i c_j}{|c|^2}\frac{\mathrm{d}\bar{c}^j \wedge \mathrm{d}c_i}{2\mathrm{i}}\right].$$

It is zero inside $|c|^2 < 1$ and has four null directions on the boundary.

Contracting the hole

Coordinate change:

$$w_i = c_i \sqrt{\frac{|c|^2 - 1}{|c|^2}}$$

Shrinks sphere $|c|^2 = 1$ to the point $|w|^2 = 0$. We get:

$$\omega = \frac{2N}{1 + |w|^2} \left(\frac{\mathrm{d}\overline{w}^i \wedge \mathrm{d}w_i}{2\mathrm{i}} - \frac{w_i \overline{w}^j}{1 + |w|^2} \frac{\mathrm{d}\overline{w}^i \wedge \mathrm{d}w_j}{2\mathrm{i}} \right)$$

This is precisely $(2\pi N)\omega_{\mathrm{FS}}$ on $\mathbb{CP}^3$!

Summary

- The Phase space is $\mathbb{CP}^{n_C - 1}$
- The symplectic form is cohomologically $(2\pi N)\omega_{\mathrm{FS}}$
- The coordinates have U(3) charges (n_1, n_2, n_3)
- This is isomorphic to the 3D harmonic oscillator

This gives the partition function

$$\sum_N \zeta^N Z_N(\mu_1, \mu_2, \mu_3) = \prod_{\vec{n}} \frac{1}{1 - \zeta \mathrm{e}^{-\mu_i n_i}}$$

Conclusions and future directions

We get exact (finite N) matching between the gauge theory and giant gravitons.

We are getting ordinary gravitons by quantising D-branes.

Giants and dual giants count the same states.

[Mandal,Suryanarayana]

Should be extended to include worldvolume gauge fields and fermions.

Can be extended to other AdS/CFT duals.

Classical 1/16 BPS giants are known, but much harder to quantise. [Kim,Lee]

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